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The Shallow Water Equations

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SUMMARY

In this paper a set of exact nonlinear equations is derived for gravity flows. By assuming the flow to be shallow and assuming the vertical acceleration to be small these equations reduce to the classical equations for long waves in shallow water. If only shallowness is assumed a set of equations results, which admits in the steady case periodic solutions for Froude numbers smaller than 1 and laminar jumps for Froude numbers larger than 1. In the last section potential flows are discussed.

1. Introduction

Of all the oscillatory phenomena in nature, water waves are perhaps the most immediately accessible to ordinary experience. Yet, theoretical elucidations of the subject are enormously complicated and profound. The theory of *long waves* is one of several approximate theories created to relieve some of the complication.

Long wave theory is based on two equations of the form

$$A(u, \eta) = 0\,, \tag{1.1}$$

where A is a differential operator, u is a certain horizontal velocity, and $y = \eta(x)$ is the equation of the free surface. These equations are, of course, only approximate, and, in trying to illuminate certain portions of the theory, I was led to ask for an exact expression for $A(u, \eta)$.

Now, the approximate equation (1.1) is normally derived [1] by assuming that the pressure in the fluid is everywhere equal to the hydrostatic pressure. When one has an exact expression for $A(u, \eta)$, however, it is possible to ask when $A(u, \eta)$ is small (and, therefore, when (1.1) is approximately correct) without making the *ad hoc* hydrostatic pressure hypothesis. By this means, one can gain some insight into the question of when the long wave theory is valid.

The question is interesting partly because doubt has been cast on the validity of the theory [2] since it predicts that the only steady flow in water of constant depth is the uniform one, a result that is known to be false. Although this particular problem has been resolved by Friedrichs' proof [3, 4] that the long wave equations are merely the first terms in a formal expansion in a parameter representing the depth, residual doubt about the theory remains partly because it is not known, in general**, when the expansion converges.

There are two long wave equations (1.1), corresponding to the two unknowns u and η. Therefore, there are really two expressions $A(u, \eta)$, and the hydrostatic pressure hypothesis must somehow imply that *both* are small. From our point of view, however, from which we ask simply when the expressions $A(u, \eta)$ are small, the long wave equations appear as the consequence of two distinct hypotheses. The first is that the water is *shallow*, which, for now, we may take to mean that the flow is a member of a one parameter family of flows whose depth goes uniformly to zero with the parameter. The second hypothesis is that the vertical acceleration is small.

If both these hypotheses are made, the long wave equations result. But since they are distinct, it is fair to ask what happens when only one is assumed and not the other. By assuming that the water is shallow but not that the vertical acceleration is particularly small, we obtain a new

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** In special cases, the expansion is known to converge. See [5].

set of equations that we shall call the *shallow water equations* to distinguish them from the long wave equations (1.1).

If the velocities are small enough, one can linearize both the shallow water equations and the long wave equations. The two sets of *linearized* equations are identical. This is a conclusion that can also be drawn from my earlier paper [6] where, among other things, it was shown that the linearized long wave equations can be derived by the argument just described from the full linearized equations in which no long wave assumption appears.

In this paper, we derive the shallow water equations in the nonlinear case and proceed to study some of their simplest consequences. In particular, we show that, unlike the long wave equations, the shallow water equations describe behavior much like the known behavior of water waves. In particular, we show that in water of constant depth the shallow water equations possess a periodic solution if the speed is subcritical. At the critical speed, the only flow is the uniform one. If the speed is supercritical, we point out that the hypothesis that the water is everywhere shallow is necessarily violated, unless the speed approaches critical speed as the depth upstream goes to zero.

In this last case, the shallow water equations possess a solution that can be interpreted as an hydraulic jump. It is like a jump in that the height of the free surface goes from one constant value to another, but the flow is smooth through the transition region. This fact will lead us to posit the existence of a laminar hydraulic jump that should occur at slightly supercritical speeds. It would be worth the effort involved to obtain experimental confirmation of the existence of such a jump.

In section 2, we derive the exact expressions for the quantities $A(u, \eta)$ appearing in the long wave equations. In section 3, the shallow water and the long wave equations are derived, and the sense in which the water is supposed to be shallow is more carefully explained. Sections 4 and 5 are devoted to flows in water of constant depth in the subcritical and supercritical cases, respectively.

At no point through section 5 will it matter to us whether the flow is irrotational or not. In many cases, however, this simplifying assumption is made, and in section 6, we derive a consequence for the shallow water equations. This consequence will be called the *consistency relation*, and it is used in section 7 to derive a first integral of the shallow water equations when the flow is steady and irrotational.

2. Some Exact Formulas for Gravity Flows

Throughout this paper, we shall be considering the flow of a non-viscous, incompressible fluid subject only to the force of gravity. In this section, we shall derive some exact formulas valid for any such flow. The quantities appearing on one side of these formulas are exactly the same as the quantities that are equal to zero in the long wave equations.

To describe the situation, we introduce a rectangular coordinate system with the y-axis directed upward and the xz-plane horizontal. Let (U, V, W) denote the velocity of the fluid at any point, let ρ denote the (constant) density of the fluid, and let P denote the pressure. Then the equations describing the motion are

$$\begin{aligned} U_t + UU_x + VU_y + WU_z &= -\frac{1}{\rho} P_x , \\ V_t + UV_x + VV_y + WV_z &= -\frac{1}{\rho} P_y - g , \\ W_t + UW_x + VW_y + WW_z &= -\frac{1}{\rho} P_z , \end{aligned} \tag{2.1}$$

and

$$U_x + V_y + W_z = 0 . \tag{2.2}$$

We want to describe the motion of the fluid over a fixed bottom which we assume to be a surface having an equation of the form $y = -h(x, z)$. Since we wish there to be no flow through this surface, we require that the normal component of the velocity be zero there:

$$V + Uh_x + Wh_z = 0 \text{ when } y = -h. \tag{2.3}$$

Assume there is a free surface described by an equation $y = \eta(x, z, t)$. Differentiating this equation totally with respect to t, we find that we must require

$$V = \eta_t + U\eta_x + W\eta_z \text{ when } y = \eta. \tag{2.4}$$

(2.4) is one of the two conditions required at the free surface. The other is that the pressure be constant there:

$$P = \text{constant when } y = \eta. \tag{2.5}$$

Equations (2.1–5) are the usual equations of water waves (see [1] and [4]). To analyze them further, let u, v and w denote the value of U, V and W *at the free surface*:

$$\begin{aligned} u(x, z, t) &= U(x, \eta, z, t), \\ v(x, z, t) &= V(x, \eta, z, t), \\ w(x, z, t) &= W(x, \eta, z, t). \end{aligned} \tag{2.6}$$

Notice that since each of the lower case letters u, v and w represents one of the components of the velocity evaluated at the free surface $y = \eta(x, z, t)$, u, v and w *do not depend on* y.

We have immediately from (2.4) that

$$v = \eta_t + u\eta_x + w\eta_z. \tag{2.7}$$

It will be convenient in what follows to let ∇ denote the *two-dimensional* gradient:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right),$$

and to write

$$\boldsymbol{u} = (u, w).$$

With this notation, (2.7) becomes

$$v = \eta_t + \boldsymbol{u} \cdot \nabla\eta. \tag{2.8}$$

Differentiating (2.6), we find the following relations to be valid:

$$\begin{aligned} u_t &= (U_t + \eta_t U_y)|_{y=\eta}, \\ u_x &= (U_x + \eta_x U_y)|_{y=\eta}, \\ u_z &= (U_z + \eta_z U_y)|_{y=\eta}. \end{aligned}$$

Consider

$$\begin{aligned} u_t + uu_x + wu_z &= [U_t + UU_x + WU_z + U_y(\eta_t + U\eta_x + W\eta_z)]_{y=\eta} \\ &= [U_t + UU_x + WU_z + VU_y]_{y=\eta} \\ &= -\frac{1}{\rho} P_x|_{y=\eta}. \end{aligned} \tag{2.9}$$

In a similar way, one can show that

$$v_t + uv_x + wv_z = -\frac{1}{\rho} P_y|_{y=\eta} - g, \tag{2.10}$$

and

$$w_t+uw_x+ww_z=-\frac{1}{\rho}P_z\Big|_{y=\eta}\,. \tag{2.11}$$

On the other hand, differentiating (2.5) first with respect to x and then with respect to z, we obtain

$$(P_x+\eta_x P_y)|_{y=\eta}=0\,,$$

and

$$(P_z+\eta_z P_y)|_{y=\eta}=0\,.$$

Inserting the values of P_x, P_y, and P_z given by (2.9–11) into these last two equations, we find that

$$u_t+uu_x+wu_z+\eta_x(g+v_t+uv_x+wv_z)=0$$
$$w_t+uw_x+ww_z+\eta_z(g+v_t+uv_x+wv_z)=0$$

or, in the notation we introduced before,

$$\boldsymbol{u}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+(g+v_t+\boldsymbol{u}\cdot\nabla v)\nabla\eta=0\,. \tag{2.12}$$

We turn now to the boundary condition (2.3). To write it in terms of the quantities η, v, and $\boldsymbol{u}$, notice that

$$\begin{aligned} U|_{y=-h}&=u-\int_{-h}^{\eta}U_y\,dy\\ W|_{y=-h}&=w-\int_{-h}^{\eta}W_y\,dy\,. \end{aligned} \tag{2.13}$$

There is a similar formula for $V|_{y=-h}$; integrating this formula once by parts, we find that

$$\begin{aligned} V|_{y=-h}&=v-(h+\eta)V_y|_{y=\eta}+\int_{-h}^{\eta}(h+y)V_{yy}\,dy\\ &=v-(h+\eta)V_y|_{y=\eta}-\int_{-h}^{\eta}(h+y)(U_{xy}+W_{zy})\,dy\,, \end{aligned} \tag{2.14}$$

by (2.2).

We substitute the expressions (2.13) and (2.14) into (2.3). This gives

$$v+uh_x+wh_z=(h+\eta)V_y|_{y=\eta}+\int_{-h}^{\eta}\left\{\frac{\partial}{\partial x}[(h+y)U_y]+\frac{\partial}{\partial z}[(h+y)W_y]\right\}dy\,.$$

By (2.7),

$$\begin{aligned} \eta_t+u(h+\eta)_x+w(h+\eta)_z&=(h+\eta)V_y|_{y=\eta}+\int_{-h}^{\eta}\left\{\frac{\partial}{\partial x}[(h+y)U_y]+\frac{\partial}{\partial z}[(h+y)W_y]\right\}dy\\ &=(h+\eta)[V_y-\eta_x U_y-\eta_z W_y]_{y=\eta}+\frac{\partial}{\partial x}\int_{-h}^{\eta}(h+y)U_y\,dy\\ &\quad+\frac{\partial}{\partial z}\int_{-h}^{\eta}(h+y)W_y\,dy\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \eta_t+[u(h+\eta)]_x+[w(h+\eta)]_z&=(h+\eta)[V_y+u_x+w_z-\eta_x U_y-\eta_z W_y]_{y=\eta}\\ &\quad+\frac{\partial}{\partial x}\int_{-h}^{\eta}(h+y)U_y\,dy+\frac{\partial}{\partial z}\int_{-h}^{\eta}(h+y)W_y\,dy\,. \end{aligned} \tag{2.15}$$

By (2.2), the first term on the right of this equation is equal to

$$(h+\eta)[u_x+w_z-(U_x+\eta_x U_y)-(W_z+\eta_z W_y)]_{y=\eta}\,,$$

and this is zero by definition of u and w. Thus, if we let $\boldsymbol{U}$ denote the horizontal component of the velocity at any point in the fluid:

$$\boldsymbol{U} = (U, W),$$

(2.15) gives

$$\eta_t + \nabla \cdot [\boldsymbol{u}(h+\eta)] = \nabla \cdot \int_{-h}^{\eta} (h+y)\boldsymbol{U}_y dy . \tag{2.16}$$

We summarize what has been derived in this section. *Let (U, V, W) be the velocity of a free surface flow under gravity. Let*

$$\boldsymbol{U} = (U, W),$$

and let

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right).$$

Let u, v and w denote the values of U, V and W on the free surface $y = \eta$, and let

$$\boldsymbol{u} = (u, w).$$

Then, the quantities u, v, w and η are related by the equations

$$\begin{aligned} v &= \eta_t + \boldsymbol{u} \cdot \nabla\eta , \\ \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &+ (g + v_t + \boldsymbol{u} \cdot \nabla v)\nabla\eta = 0 , \\ \eta_t + \nabla \cdot [\boldsymbol{u}(h+\eta)] &= \nabla \cdot \int_{-h}^{\eta} (h+y)\boldsymbol{U}_y dy . \end{aligned} \tag{2.17}$$

Equations (2.17) form the basis for everything that follows.

3. The Approximate Equations

It is instructive at this point to write down the classical long wave equations. They have the form [4]

$$\begin{aligned} &\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + g\nabla\eta = 0 , \\ &\eta_t + \nabla \cdot [\boldsymbol{u}(h+\eta)] = 0 . \end{aligned} \tag{3.1}$$

Comparing these equations with the second and third of equations (2.17), we arrive at the following criterion for the validity of the long wave equations:

The long wave equations (3.1) *may be expected to be valid in any flow in which*

$$|v_t + \boldsymbol{u} \cdot \nabla v| \ll g \tag{3.2}$$

and, in addition, in which

$$\left|\int_{-h}^{\eta} (h+y)\boldsymbol{U}_y dy\right| \ll \left|\boldsymbol{u}(h+\eta)\right| . \tag{3.3}$$

It will be our purpose in this section to interpret conditions (3.2) and (3.3) physically to give a simple criterion for the validity of (3.1). After that, it will be possible simply to write down the shallow water equations and to state a criterion for their validity.

The interpretation of (3.2) is immediate. The vertical velocity of a particle on the free surface is

$$\begin{aligned} \frac{d\eta}{dt} &= \eta_t + \boldsymbol{u} \cdot \nabla\eta \\ &= v , \end{aligned}$$

and the vertical acceleration of such a particle is

$$\frac{d^2\eta}{dt^2} = v_t + \boldsymbol{u}\cdot\nabla v\,.$$

Since this is the left side of (3.2), we see that (3.2) *amounts to the condition that the vertical acceleration of a particle on the free surface is small compared with the acceleration due to gravity.*

(3.3) can be valid under various different hypotheses. We shall refer to (3.3) however as the *shallow water condition.* To see why, consider the left side of (3.3). $\boldsymbol{U}_y$ is the rate of change of the horizontal velocity in the vertical direction. If a denotes a typical acceleration and s a typical speed in a flow, $\boldsymbol{U}_y$ is of the order of a/s. Thus, the left side of (3.3) is of order

$$\frac{a}{s}\int_{-h}^{\eta}(h+y)\,dy = \frac{a}{s}\frac{(h+\eta)^2}{2}\,,$$

and (3.3) will hold if this is of a smaller order of magnitude than $\boldsymbol{u}(h+\eta)$. But $h+\eta$ is obviously the local depth. Therefore if δ is a typical depth, (3.3) will hold if

$$\frac{a\delta}{s^2} \ll 1\,. \tag{3.4}$$

This means, in particular, that if we have a family of flows in which the acceleration and the velocity are uniformly bounded while the depth goes uniformly to zero, (3.4) will surely be satisfied. A flow in which (3.4) holds may therefore legitimately be called a *shallow water flow.*

Notice that in a shallow water flow, the acceleration may be small, in which case the depth need not be. On the other hand, if the depth is small, the acceleration need not be small. Therefore, shallow water flows are governed by the equations obtained from (2.17) by assuming (3.3) but not (3.2), that is, by the equations

$$\begin{aligned} &v = \eta_t + \boldsymbol{u}\cdot\nabla\eta\,, \\ &\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (g + v_t + \boldsymbol{u}\cdot\nabla v)\nabla\eta = 0\,, \\ &\eta_t + \nabla\cdot[\boldsymbol{u}(h+\eta)] = 0\,. \end{aligned} \tag{3.5}$$

Equations (3.5) will be referred to as the *shallow water equations.*

4. Water of Constant Depth

If h is constant, we choose the coordinate system so that the plane $y=0$ coincides with the bottom. If, in addition, we suppose the flow to be steady and two-dimensional, we can take $w=0$, assume everything independent of z, and reduce (3.5) to the form

$$v = u\eta_x\,,$$

$$uu_x + (g + uv_x)\eta_x = 0\,,$$

$$\frac{\partial}{\partial x}(u\eta) = 0\,.$$

These equations can be integrated easily to show that

$$\begin{aligned} &g\eta + \tfrac{1}{2}u^2(1+\eta_x^2) = \frac{c^2}{2}\,, \\ &u\eta = Q\,, \end{aligned} \tag{4.1}$$

where c^2 and Q are constants. Clearly, we must always require

$$\eta \geqq 0 \tag{4.2}$$

since in our coordinate system $h=0$; thus, c^2 is positive, as the notation indicates.

Eliminating u from equations (4.1), we obtain the following ordinary differential equation for the free surface in two-dimensional, steady, shallow water flow in water of constant depth:

$$Q^2(1+\eta_x^2) = (c^2-2g\eta)\eta^2 \,. \tag{4.3}$$

Equation (4.3) can be solved explicitly for η as a function of x, and we shall solve it shortly. However, it is more illuminating to study (4.3) geometrically by means of a "phase-plane" diagram of η_x *versus* η. The gross features of this diagram can be described immediately, since η_x^2 is a cubic in η. Thus, the result must be as in Figure 1, which we now proceed to interpret.

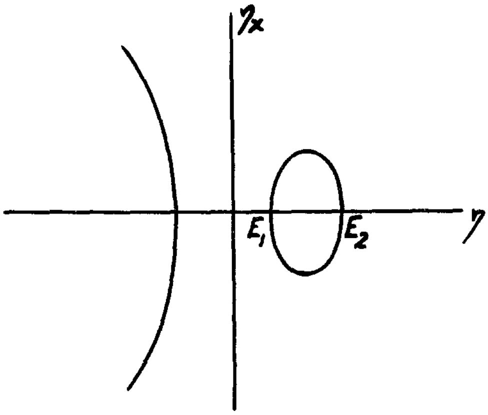

Figure 1.

The left-hand branch of the figure may be ignored entirely since, as we shall see, on that branch (4.2) is always violated. As for the right-hand branch, the points on the η-axis marked E_1 and E_2 are equilibrium points. They correspond to *constant* solutions for which $\eta_x=0$.

In addition to these constant solutions, we have a periodic solution beginning at any point on the right-hand branch that moves clockwise around the branch and repeating itself once it comes back to the initial point.

All this can be said without going into a detailed study of (4.3). We shall now consider more carefully the periodic solution represented by the oval in Figure 1. To do so, it is convenient to describe the motion by means of parameters other than the constants c^2 and Q of equations (4.3).

Let η_0 be the maximum value of η, and let u_0 be the velocity at the point where the maximum occurs. Then, according to (4.1b),

$$Q = u_0\eta_0 \,.$$

Also, since η_0 is a maximum, η_x is zero at the point where η_0 occurs. Therefore, (4.1a) gives

$$c^2 = u_0^2+2g\eta_0 \,.$$

Substituting these constants into (4.3), we obtain an equation that can be written in the form

$$u_0^2\eta_0^2\eta_x^2 = (\eta_0-\eta)[2g\eta^2-u_0^2(\eta_0+\eta)] \,. \tag{4.4}$$

The right-hand side is zero when $\eta=\eta_0$, which is as it should be. $\eta=\eta_0$ is the right-hand equilibrium point E_2 of Figure 1. We can now see directly from (4.4) that $\eta\equiv\eta_0$ is a solution.

Let

$$F = \frac{u_0^2}{g\eta_0} \tag{4.5}$$

be the Froude number of the flow. We shall call the flow *subcritical* if $F<1$, *critical* if $F=1$, and *supercritical* if $F>1$. The supercritical case will be discussed in the next section; for now, we assume $F\leqq 1$.

One zero of the right side of (4.4) is $\eta=\eta_0$. The others are

$$\eta=\eta_0\left[\frac{F}{4}\pm\tfrac{1}{4}(F^2+8F)^{\frac{1}{2}}\right]. \tag{4.6}$$

The lower sign corresponds to a negative value of η. Therefore, as we said earlier, the left-hand branch of Figure 1 corresponds entirely to negative values of η and may be ignored. The upper sign in (4.6) corresponds to the equilibrium point E_1 of Figure 1. If $F \leqq 1$, E_1 occurs to the left of E_2, as it should if η_0 is a maximum of η. Thus we see that whenever the flow is subcritical, there is a periodic solution of the steady shallow water equations. This is as it should be, for it is known that if $F<1$, a periodic flow exists.

When $F\to 1$, $E_1\to E_2$, as (4.6) shows. Therefore, in the critical case, the only shallow water flow is the trivial one $\eta=\eta_0$ corresponding to the uniform velocity $u=u_0$.

Finally, we note that (4.4) can be integrated to obtain further details of the flow. The result is

$$x-x_0=u_0\eta_0\int\frac{d\eta}{\{(\eta_0-\eta)[2g\eta^2-u_0^2(\eta_0+\eta)]\}^{\frac{1}{2}}}, \tag{4.7}$$

where x_0 is constant. This integral can be evaluated in terms of inverse elliptic functions. We shall not bother to carry out the details.

5. The Laminar Jump

In this section, we continue to assume $h\equiv 0$ and study the supercritical case when the Froude number F exceeds unity. In that case, the flow is unstable [7]. Therefore, any flow that is uniform and supercritical far upstream is likely to change over to another, stable regime. This other regime is often also uniform, but it is subcritical and, therefore, stable. The transfer from one regime to another takes place over a rather short distance and is called a *hydraulic jump*.

Such a jump is usually considered to be a discontinuous solution of the long wave equations, with appropriate conditions across the jump being added on. But this entails a strange situation. The long wave theory requires that the vertical accelerations be small. However, this can never be the case near a jump, where the velocity changes discontinuously. How, then, can the long wave theory explain the jump? Presumably, the difficulty is somehow taken care of automatically by the jump conditions, although exactly how this happens is obscure.

One might hope that with the shallow water theory, which involves no assumption on the acceleration, the hydraulic jump might be explained directly, without the intervention of additional jump conditions, and even that the structure of the flow within the jump might be described. Unfortunately, we cannot quite achieve all this, no doubt because of energy losses that occur in most actual jumps.

However, for Froude numbers less than two, the energy loss in a jump is very small. (Indeed, according to the curve in [9, Figure 8], *the energy loss is strictly zero for* $F<1.8$.) In view of this fact, one might suspect that for Froude numbers less than, say 1.8, the usual explanation of the hydraulic jump is faulty, involving, as it always does, a loss in energy. We shall be led to this same conclusion from the shallow water theory which, as we shall see, predicts the existence of a jump, but one which, unlike the usual hydraulic jump, is laminar and which, therefore, takes place without loss of energy.

The existence of this laminar jump must probably be left for experiment to determine. To aid in this task, we describe the flow through the jump, examine briefly the length and magnitude of the jump, and show that the magnitudes of the hydraulic and the laminar jumps are asymptotically the same as the Froude number goes to unity.

However, we begin by showing that it would be too much to expect that the shallow water theory explain the hydraulic jump in general since such jumps are usually not shallow in our sense. Consider a two-dimensional, uniform flow, beginning at $x=-\infty$ with speed u_0 and depth η_0. We suppose that

$$F_0 = \frac{u_0^2}{g\eta_0} > 1 \, .$$

As we just said, such a flow is unstable and is likely to jump to another uniform flow with speed u_1^* and depth η_1^*, say. Conservation of mass and momentum across the jump imply that u_1^* and η_1^* are obtained from u_0 and η_0 by the formulas [8]:

$$\eta_1^* = \eta_0 \frac{-1+(1+8F_0)^{\frac{1}{2}}}{2} \, , \tag{5.1}$$

and

$$u_1^* = u_0 \frac{1+(1+8F_0)^{\frac{1}{2}}}{4F_0} \, . \tag{5.2}$$

We consider a family of flows in which $\eta_0 \to 0$. An interesting example occurs when the discharge Q is given. In that case, $u_0 = Q/\eta_0$, and the flow is shallow near $x = -\infty$ since, taking $\delta = \eta_0$ and $s = u_0$ in (3.4), we find that that condition becomes

$$\frac{a}{Q^2} \eta_0^3 \ll 1 \, ,$$

and this is correct since $\eta_0 \to 0$.

We have $F_0 = Q^2/g\eta_0^3 \to \infty$. Therefore, (5.1) and (5.2) give

$$\eta_1^* \sim \left(\frac{2Q^2}{g\eta_0}\right)^{\frac{1}{2}} \, ,$$

$$u_1^* \sim \left(\frac{g\eta_0}{2}\right)^{\frac{1}{2}} \, .$$

We wish to investigate whether the flow can be shallow to the right of the jump. To do so, set $\delta = \eta_1^*$, $s = u_1^*$ in (3.4). We find that the flow is shallow if

$$\frac{aQ}{(g\eta_0)^{\frac{3}{2}}} \ll 1 \, . \tag{5.3}$$

But $\eta_0 \to 0$. Therefore, (5.3) cannot be satisfied for fixed Q unless the acceleration goes to zero; and even more, goes to zero faster than $\eta_0^{\frac{3}{2}}$. Since η_0 is approaching zero, however, while η_1^* actually grows, the jump must be extremely turbulent, and it is hard to make any predictions whatever about a.

All of this assumes that Q is fixed, so that u_0 varies like $1/\eta_0$. If u_0 goes to zero with η_0 in such a way that $F_0 \to 1$, on the other hand, things are different. In that case,

$$\eta_1^* \sim \eta_0 \, ,$$

$$u_1^* \sim u_0 \, ,$$

and the flow is shallow to the right of the jump since it is shallow to the left.

Therefore, we consider again the steady, two-dimensional, shallow water equations which, as before, can be written in the form (4.4). In the present context, the parameters u_0 and η_0 occurring in (4.4) are to be interpreted as the velocity and the depth infinitely far upstream.

The phase plane diagram remains the same as in Figure 1, but the roles of E_1 and E_2 are reversed. When $F_0 > 1$, E_1 is the (unstable) equilibrium point $(\eta_0, 0)$. E_2 is the point $(\eta_1, 0)$, where

$$\eta_1 = \eta_0 \left[\frac{F_0}{4} + \tfrac{1}{4}(F_0^2 + 8F_0)^{\frac{1}{2}} \right] \, . \tag{5.4}$$

The motion can now be described as follows. It begins at the equilibrium point E_1. Since E_1 is unstable, the point describing the motion will tend to leave E_0 and move along the oval curve of Figure 1. Since η must increase when the point leaves E_1, η_x must be positive. Therefore, the point that was originally at E_1 will, upon meeting any slight disturbance, move away from E_1 along the upper half of the oval curve of Figure 1, until it reaches E_2. Since E_2 is an equilibrium point, the motion can stop there and, since E_2 is stable, once having done so, the motion will remain uniform. The motion we have just described will be referred to as a *laminar jump.*

It is important to notice that the height of this laminar jump is different from the height of the hydraulic jump, as a quick comparison of (5.1) and (5.4) shows. However,

$$\frac{\eta_1^*}{\eta_1} = 2\,\frac{-1+(1+8F_0)^{\frac{1}{2}}}{F_0+(F_0^2+8F_0)^{\frac{1}{2}}} \to 1 \tag{5.5}$$

as $F_0 \to 1$. That the two heights should differ is not surprising, since the hydraulic jump involves an energy loss through the dissipation of heat, while the laminar jump, being laminar and non-viscous, involves no such loss.

If this explanation is correct, we should expect the height of the laminar jump to exceed that of the hydraulic jump. And that is the case, since the quantity on the right of (5.5) is always less than one, as a simple calculation shows.

Unfortunately, it is difficult to distinguish between the two kinds of jump on the basis of height alone since, if it exists at all, the laminar jump will only exist near $F=1$, while the difference in the height of the jumps is order $(F-1)^2$. Indeed, (5.5) shows that $\eta_1^*/\eta_1 = 1+0((F-1)^2)$ as $F \to 1$. On the other hand, the laminar jump does have a unique *length*, defined as the distance L required for the depth to move from η_0 to η_1. L can be computed from (4.7). If the jump starts at $x=x_0$ and ends at $x=x_1$, then $L=x_1-x_0$, while η is η_0 at x_0 and η_1 at x_1. Therefore,

$$L = u_0\eta_0 \int_{\eta_0}^{\eta_1} \frac{d\eta}{\{(\eta_0-\eta)[2g\eta^2-u_0^2(\eta_0+\eta)]\}^{\frac{1}{2}}},$$

where η_1 is given by (5.4). Writing $\eta=\eta_0\xi$ in this integral, we reduce it to the simpler form

$$L = \eta_0 F_0^{\frac{1}{2}} \int_1^{\eta_1/\eta_0} \frac{d\xi}{\{(1-\xi)(2\xi^2-F_0\xi-F_0)\}^{\frac{1}{2}}}. \tag{5.6}$$

Since η_1/η_0 depends only on the Froude number F_0, the same is true of the integral, and we conclude that for fixed F_0, L is proportional to η_0.

On the other hand, for fixed η_0, a slightly complicated but straightforward computation shows that as $F_0 \downarrow 1$,

$$L = \eta_0\left[\frac{\pi}{3^{\frac{1}{2}}} + 0\{(F_0-1)^{\frac{1}{2}}\}\right].$$

Thus, as $F_0 \downarrow 1$, the length of the laminar jump has a definite, non-zero limit. This is apparently entirely different from the behavior of the hydraulic jump. If Figure 6 of [9] is to be believed, the length of the hydraulic jump approaches zero as F_0 approaches one. However, there is some question about the validity of this figure near $F=1$, and the issue remains in doubt.

6. Potential Flow

So far, we have not assumed the flow to be irrotational. If it is irrotational, however, there is another relationship between η and $\boldsymbol{u}$, in addition to (2.17), that is fulfilled. We shall derive that relation here.

If the flow is irrotational, there is a potential Φ such that

$$U=\Phi_x\,,\quad V=\Phi_y\,,\quad W=\Phi_z\,.$$

It is well known that in this case, the three equations (2.1) collapse to the single Bernoulli equation

$$gy + \frac{1}{\rho} P + \Phi_t + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) = \text{constant}\,, \tag{6.1}$$

while (2.2) becomes just Laplace's equation

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0\,. \tag{6.2}$$

Now, if we define u as before, by (2.6), we find that

$$\begin{aligned} u_z &= (U_z + \eta_z U_y)|_{y=\eta} \\ &= (\Phi_{xz} + \eta_z \Phi_{xy})|_{y=\eta}\,. \end{aligned} \tag{6.3}$$

In a similar way, the following three formulas can be proved:

$$\begin{aligned} w_x &= (\Phi_{xz} + \eta_x \Phi_{yz})|_{y=\eta} \\ v_x &= (\Phi_{xy} + \eta_x \Phi_{yy})|_{y=\eta} \\ v_z &= (\Phi_{yz} + \eta_z \Phi_{yy})|_{y=\eta} \end{aligned} \tag{6.4}$$

(6.3) and (6.4) can be used immediately to prove

$$w_x + \eta_z v_x = u_z + \eta_x v_z\,. \tag{6.5}$$

In any flow derivable from a potential, the components of the velocity *on the free surface* must be related by (6.5). Formula (6.5) will be referred to in the sequel as *the consistency relation.*

Before proceeding, two remarks about (6.5) should be made. First, we note that although it depends on the hypothesis that the flow is irrotational, it is like (2.17) in that it is not approximate.

Second, it should be noted that *for two-dimensional flows, the consistency relation degenerates to an identity.* Indeed, if the flow is two-dimensional, we may assume it takes place in planes z=constant. Then, everything is independent of z, and $w=0$. But when everything is independent of z, (3.5) becomes simply $w_x=0$. Thus, in two dimensions the condition of irrotationality imposes no restriction on η, u, and v. This is the reason we did not derive the consistency relation earlier.

7. Steady Flow

When a flow is steady and irrotational, the consistency relation (6.5) allows us to obtain a first integral of the shallow water equations. Rather than derive this integral directly from the shallow water equations, however, we shall return to the basic equations (2.1–5), derive what we want directly from there, and then prove that we have integrated the shallow water equations.

We suppose the flow to be steady and irrotational. Then, there is a potential Φ, and Bernoulli's equation (6.1) is valid. In (6.1), let $y=\eta$. Then, the boundary condition (2.5) and the hypothesis that the flow is steady give

$$g\eta + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2)|_{y=\eta} = \tfrac{1}{2}c^2\,,$$

where c is a constant having the dimensions of a velocity. On the other hand, we have immediately from the definitions of u, v, and w that

$$\Phi_x|_{y=\eta} = u, \quad \Phi_y|_{y=\eta} = v, \quad \Phi_z|_{z=\eta} = w\,.$$

Therefore, we find

$$g\eta + \tfrac{1}{2}(u^2 + v^2 + w^2) = \frac{c^2}{2}$$

or

$$g\eta + \tfrac{1}{2}(|\boldsymbol{u}|^2 + v^2) = \frac{c^2}{2}\,, \tag{7.1}$$

where c is a constant.

We shall now show that (7.1) and the consistency relation are completely equivalent to the shallow water equation (3.5b). Differentiating the left-hand side of (7.1) and using the consistency relation, we find

$$\begin{aligned}\frac{\partial}{\partial x}\left[g\eta+\tfrac{1}{2}(|\boldsymbol{u}|^2+v^2)\right] &= g\eta_x+uu_x+vv_x+ww_x\\ &= \boldsymbol{u}\cdot\nabla u+\eta_x(g+wv_z)+v_x(v-\eta_z w)\\ &= \boldsymbol{u}\cdot\nabla u+\eta_x(g+\boldsymbol{u}\cdot\nabla v)\,,\end{aligned} \tag{7.2}$$

by definition, (2.8), of v. In a similar way, differentiation of (7.1) with respect to z gives

$$\frac{\partial}{\partial z}\left[g\eta+\tfrac{1}{2}(|\boldsymbol{u}|^2+v^2)\right] = (\boldsymbol{u}\cdot\nabla)w+\eta_z(g+\boldsymbol{u}\cdot\nabla v)\,. \tag{7.3}$$

An immediate consequence of (7.2) and (7.3) is that (7.1) is equivalent to (3.5b) whenever the consistency relation holds and the flow is steady.

Thus, we see that whenever the flow is steady and irrotational, the shallow water equations can be replaced by the following equivalent but simpler set of equations:

$$\begin{aligned}&v=\boldsymbol{u}\cdot\nabla\eta\,,\\ &2g\eta+(|\boldsymbol{u}|^2+v^2)=c^2\,,\\ &\nabla\cdot[\boldsymbol{u}(h+\eta)]=0\,,\end{aligned} \tag{7.4}$$

along with the consistency relation (6.5).

It is worth remarking, finally, that of the four steady shallow water equations (6.5) and (7.4), only (7.4c) is approximate. (7.4a, b) and (6.5) are direct consequences of the exact equations (2.1–5) and the hypothesis that the flow is irrotational.

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